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Stability of a vortex with a heavy core

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This paper examines the stability of swirling flows in a non-homogeneous fluid. Density gradients are shown to produce two distinct kinds of instability. The first is the centrifugal instability (CTI) which mainly affects axisymmetric, short-axial-wavelength eigenmodes. The second is the Rayleigh–Taylor instability (RTI) which mainly affects non-axisymmetric, two-dimensional eigenmodes. These instabilities are described for a family of model flows for which the velocity law $V(r)$ corresponds to a Gaussian vortex with radius 1, and the density law $R(r)$ corresponds to a Gaussian distribution characterized by a density contrast C and a characteristic radius b. A full map in the (C, b)-plane is given for the amplification rate and the structure of the most amplified eigenmode. For small density contrasts ($C < 0.5$), the CTI occurs only for $b > 1$ and the RTI for $b \lesssim 0.8$. On the other hand, for high density contrasts ($C > 0.5$), a competition between the two kinds of instabilities is observed. From a fundamental point of view, the nature of the instability depends on the local values of $G^2 = -r^{-1}V^2R^{-1}\mathrm{d}R/\mathrm{d}r$ and the Rayleigh discriminant $\Phi = r^{-3}\mathrm{d}(r^2V^2)/\mathrm{d}r$. CTI occurs whenever $G^2 > \Phi$ somewhere in the flow. For RTI, a necessary condition is that $G^2 > 0$ somewhere in the flow. By an asymptotic analysis, we show that this condition is also sufficient in the limit $b \to 0$, $C \to 0$. This asymptotic analysis also confirms that shear has a stabilizing effect on RTI and that this instability is strictly analogous to the standard RTI obtained in the case where light fluid is situated below heavier fluid in the presence of gravity.

1. Introduction

Stabilizing mechanisms associated with rotation usually make a vortex very resistant to radial momentum diffusion. The present paper considers possible density variation effects to achieve this. If density effects were significant, vortex control by means of injection of heated or cooled air could be considered for example in application to aircraft wakes. The precise goal of this paper is to evaluate the potential of such density effects to produce linear instabilities.

We focus on a non-homogeneous fluid which is driven by the equations of motion, incompressibility and continuity:

$$\rho(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) = -\nabla p, \quad \nabla\cdot\boldsymbol{u} = 0, \quad \partial_t\rho + \boldsymbol{u}\cdot\nabla\rho = 0, \tag{1.1}$$

where $\boldsymbol{u}$, p and ρ stand for the velocity, pressure and density. In cylindrical coordinates (r, θ, z), let u, v and w be the radial, azimuthal and axial components of the

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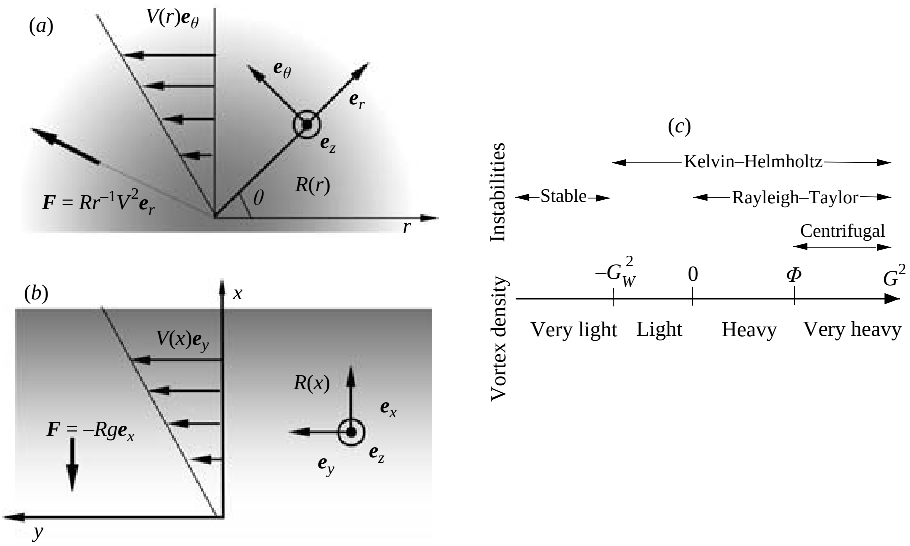

FIGURE 1. (*a*) Non-homogeneous swirling flow. (*b*) Non-homogeneous shear flow under the action of gravity. The grey levels show the density field. The same notation has been chosen on purpose in both settings to show their analogies. (*c*) Overview of instabilities in the case where $\Phi > 0$.

velocity field. As shown in figure 1(*a*), we choose a steady basic flow of the form $[u, v, w, p, \rho] = [0, V(r), 0, P(r), R(r)]$ where $V(r)$ and $R(r)$ are two given functions which characterize the azimuthal velocity and the density of the vortex. The pressure $P(r)$ equilibrates the centrifugal force $\boldsymbol{F} = Rr^{-1}V^2\boldsymbol{e}_r$ so that $P' = Rr^{-1}V^2$ where the prime denotes differentiation with respect to the radial coordinate r. We are particularly interested in the case where a vortex has a heavy internal core so that $R' < 0$ for all radii.

In the framework of non-homogeneous swirling flow stability, extensive work has already uncovered two important quantities: the Rayleigh discriminant $\Phi = 2r^{-1}V\Xi$, where $\Xi = V' + r^{-1}V$ is the vorticity of the basic flow, and $G^2 = -r^{-1}V^2R^{-1}R'$. Note that $-G^2$ corresponds, if $R' > 0$, to the square of the buoyancy frequency, which is analogous to the standard buoyancy frequency (or Brunt–Väisälä frequency) with gravity replaced by the centrifugal acceleration $r^{-1}V^2$ (see § 4 for more details on the analogy).

Figure 1(*c*) gives an overview of the instabilities that may occur in such flows as a function of G^2. We have the following results:

(i) If $G^2 > \Phi$ for some radius r, which corresponds to a very heavy vortex, Eckhoff (1984) showed by a Wentzel–Kramers–Brillouin (WKB) analysis that the flow is subject to centrifugal instability (CTI). Le Duc & Leblanc (1999) showed that a classical normal mode analysis with axisymmetric perturbations retrieves the results of the WKB analysis in the limit of small axial wavelengths. This case will be addressed in § 3.

(ii) If $G^2 \leqslant \Phi$ for all radii, Leibovich (1969) and Howard (1973) have shown that the flow field is stable to all axisymmetric perturbations.

(iii) If $0 < G^2 < \Phi$, which corresponds to a heavy vortex, there exists no general sufficient condition for instability. This case will thoroughly be analysed in § 4. An interesting result has been shown by Gans (1975) in the case of rigidly rotating flows where $V \sim r$: if G^2 is slightly positive then the flow is unstable to two-dimensional non-axisymmetric perturbations. This result may also be inferred for all $G^2 > 0$ from the WKB analysis of Eckhoff (1984) and the work of Le Duc (2001). We will show in this paper that this instability is actually a Rayleigh–Taylor instability (RTI).

(iv) If $G^2 = 0$, i.e. in the case of homogeneous flows $R' = 0$, Drazin & Reid (1981) showed that a necessary condition for two-dimensional non-axisymmetric instability is that the vorticity distribution Ξ presents an extremum somewhere, i.e. $\Xi' = 0$. This is a Kelvin–Helmholtz instability (KHI) which induces the rollup of annular vorticity concentrations. In the case of non-homogeneous flows with $R' \neq 0$, this necessary condition for instability does not hold anymore. Nevertheless, one may infer from the stability characteristics of a vortex sheet associated with two uniform streams of different velocities and densities in the presence of gravity (see Drazin & Reid 1981) that non-homogeneities have a destabilizing (respectively stabilizing) effect on KHI if $G^2 > 0$ (respectively $G^2 < 0$).

(v) If $G^2 < -G_W^2$ for all radii with $G_W^2 = (V' - r^{-1}V)^2/4$, Lalas (1975), Warren (1975) and Fung (1983) showed that the flow is stable to all disturbances. Note that this result is analogous to the sufficient condition for stability established by Howard (1961) which states that a standard shear flow in the presence of gravity is stable if the local Richardson number is everywhere greater than or equal to $1/4$. The case $G^2 < -G_W^2$ corresponds to a very light vortex where the stabilizing effect due to negative values of G^2 prevails over all instability mechanisms, especially KHI. Note that in the case of rigidly rotating flows, for which no KHI exists, $G_W^2 = 0$. The present sufficient condition for stability associated with the sufficent condition for instability established by Gans (1975) and Eckhoff (1984), which was mentioned above, shows that a rigidly rotating flow is unstable if and only if $G^2 > 0$ somewhere in the flow.

Note that the results of Eckhoff (1984), Le Duc & Leblanc (1999), Howard (1973), Le Duc (2001), Lalas (1975) and Warren (1975) were established in a fully compressible framework.

The main objective of this paper is to study the stability of a family of basic flows described in § 2 which represents a vortex with a heavy internal core. We are particularly interested in the competition that may exist between CTI presented in § 3 and RTI described in § 4. Note that the present article only deals with the linear regime of the perturbations. The nonlinear regime of the RTI has been addressed by Coquart, Sipp & Jacquin (2004) and Joly, Fontane & Chassaing (2004) by direct numerical simulations.

2. The basic flows and the perturbations

We study the linear stability of a family of basic flows with two parameters C and b. The velocity field consists of a Lamb–Oseen vortex of circulation $\Gamma = 2\pi$ and unitary radius. The density distribution exhibits a Gaussian-type peak in the centre of the vortex with amplitude s and width b:

$$V = r^{-1}[1 - \exp(-r^2)], \quad R = 1 + s\exp(-r^2/b^2). \tag{2.1}$$

Instead of the amplitude s, we use the density contrast parameter $C = (R_{max} - R_{min})/(R_{max} + R_{min}) = s/(2+s)$ to characterize the density distribution. We focus on the case $0 < C < 1$, which corresponds to a heavy core.

We superpose on the basic flow (2.1) small-amplitude perturbations of the form $(u, v, w, p, \rho) = [u(r), v(r), w(r), p(r), \rho(r)] \exp[\mathrm{i}(kz + m\theta - \omega t)]$ where k is the real axial wavenumber, m the azimuthal wavenumber and ω the complex frequency. Linearization of the governing equations (1.1) around the basic flow (2.1) leads to the following equation:

$$\frac{rp^2}{R}\left(\frac{R}{rp^2}\psi'\right)' - \underbrace{k^2\left(1+\frac{H^2}{\Sigma^2}\right)\psi}_{\mathrm{I}} - \underbrace{\left[l^2\left(1+\frac{G^2}{\Sigma^2}\right)+l\frac{(R\Xi)'}{R\Sigma}\right]\psi}_{\mathrm{II}} - \underbrace{\frac{2l}{r}\frac{k^2}{p^2}\frac{\Xi}{\Sigma}\psi}_{\mathrm{III}} = 0 \quad (2.2)$$

where $\psi(r) = ru(r)$, $l = mr^{-1}$, $p^2 = k^2 + l^2$, $\Sigma = -\omega + lV$ and $H^2 = G^2 - \Phi$. Equation (2.2) along with the boundary conditions $\psi(0) = \psi(\infty) = 0$ constitutes an eigenvalue/eigenvector problem for $\omega/\psi(r)$. Writing the perturbation equation in this form allows us to identify the different physical mechanisms involved: terms I and II in equation (2.2) are respectively responsible for CTI and RTI whereas term III is a coupling term which is zero for axisymmetric ($m = 0$) or two-dimensional ($k = 0$) perturbations.

The solutions to the eigenvalue/eigenvector problem (2.2) may be obtained numerically using a shooting method. Integration is achieved with a classical fourth-order Runge–Kutta scheme. At the approach to critical points r_c satisfying $\Sigma(r_c) = 0$ in the complex r-plane, the integration path is deformed according to the criterion given by Lin (1955). Preliminary results using this method were presented by Fabre *et al.* (2003).

3. Centrifugal instability

We first focus on three-dimensional ($k \neq 0$), axisymmetric ($m = 0$) eigenmodes. Equation (2.2) therefore reduces to

$$(r^{-1}R)^{-1}(r^{-1}R\psi')' - k^2\psi - k^2\omega^{-2}H^2\psi = 0. \quad (3.1)$$

This equation and its boundary conditions form a classical Sturm–Liouville eigenvalue/eigenvector problem. Hence, following Bender & Orszag (1978), if $H^2 > 0$ somewhere in the flow, then the flow is unstable. Note that the quantity H^2 can also be written $H^2 = -(Rr^3)^{-1}(Rr^2V^2)'$, so that for H^2 to be positive, the function Rr^2V^2 has to decrease somewhere in the flow.

In the limit of short-wave perturbations ($k \gg 1$), it is possible to construct analytical eigenvalues/eigenvectors following the procedure given by Bayly (1988) and Le Duc & Leblanc (1999). First, we suppose that H has a positive maximum at some given radius r_0 so that $H(r_0) = H_0 > 0$, $H'(r_0) = 0$ and $H''(r_0) = H_2 < 0$. Then, we introduce the scalings : $r = r_0 + \lambda^{1/2}k^{-1/2}\tilde{r}$ and $\omega = \mathrm{i}H_0[1 - (2\lambda)^{-1}k^{-1}\tilde{\omega}]$ with $\lambda = [-H_0(4H_2)^{-1}]^{1/2}$. If we let $k \to \infty$, equation (3.1) becomes at leading order

$$\mathrm{d}^2\psi/\mathrm{d}\tilde{r}^2 + (\tilde{\omega} - \tilde{r}^2/4)\psi = 0 \quad (3.2)$$

with $\psi \to 0$ at $\tilde{r} = 0$ and $\tilde{r} = \infty$. Following Bender & Orszag (1978), this is the quantum harmonic oscillator, so that equation (3.1) exhibits the following eigenvalues/eigenvectors in the limit $k \to \infty$:

$$\omega_n = \mathrm{i}H_0[1 - (2\lambda)^{-1}k^{-1}(n + 1/2)], \quad \psi_n = \mathrm{He}_n(\tilde{r})\exp(-\tilde{r}^2/4) \quad (3.3)$$

where He_n is the Hermite polynomial of degree n [$\mathrm{He}_0(x) = 1$, $\mathrm{He}_1(x) = x$, $\mathrm{He}_2(x) = x^2 - 1, \ldots$]. Note that n corresponds to the number of nodes of the eigenmode ψ_n. In the case of homogeneous vortices, Gallaire & Billant (2003) recently proved that CTI also exists for $m \neq 0, k \to \infty$, but the instability is less amplified than in the axisymmetric case. As will be shown in §4.2, this conclusion also holds for non-homogeneous vortices.

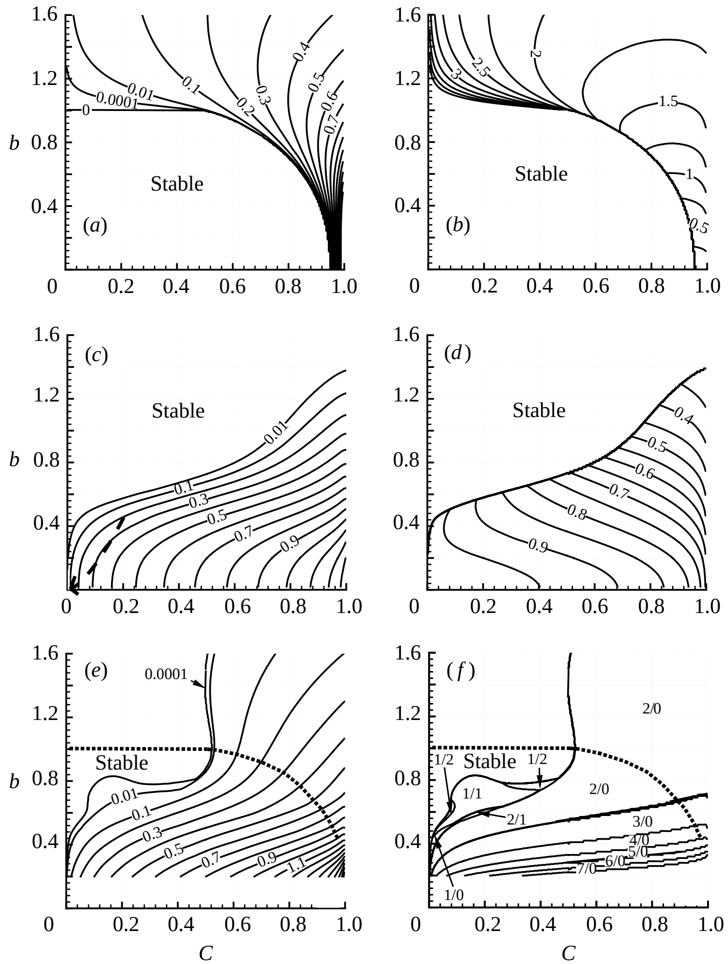

FIGURE 2. CTI with ($m=0$, $n=0$, $k\rightarrow\infty$): amplification rate ω_i (*a*) and location r_0 (*b*) of unstable eigenmode. RTI with ($m=3$, $n=0$, $k=0$): amplification rate ω_i (*c*) and frequency ω_r/m (*d*). The dashed line in plot (*c*) sketches the path $b=C^{3/2}/0.2$ (see also figure 3*b*). Most amplified RTI eigenmode for all ($m\geqslant 1$, $n\geqslant 0$, $k=0$): amplification rate ω_i (*e*) and structure m/n (*f*). In (*e*) and (*f*), a dotted line has been sketched above which the flow is unstable to CTI eigenmodes whose amplification rates are higher than those of the RTI.

Figures 2(*a*) and 2(*b*) display respectively in the (C, b)-plane the iso-values of the asymptotic amplification rate $\omega_i=H_0$ and the iso-values of the radius r_0 where the eigenmode is localized. We observe that the flow is stable for small values of C and b. There exists a non-trivial marginal stability boundary above which the flow is unstable. In the unstable region, the amplification rate is maximum for very high density contrasts C and small density radii b. In this case, the eigenmode is localized in the centre of the vortex. As b increases, the amplification rates get weaker (ω_i decreases), the eigenmodes move outward (r_0 increases), while flows with smaller values of C become unstable. For $b>1$, the flow is unstable for all values of $C>0$. But for $C<1/2$, the amplification rates are very weak and the eigenmodes are located outside the vortex core (i.e. $r_0>2$).

4. Rayleigh–Taylor instability

In this section we mainly consider two-dimensional ($k=0$) non-axisymmetric ($m \neq 0$) perturbations. Equation (2.2) therefore reduces to

$$(rR)^{-1}(rR\psi')' - l^2\psi - \underbrace{l^2\Sigma^{-2}G^2\psi}_{\text{IV}} - \underbrace{l(R\Sigma)^{-1}(R\Xi)'\psi}_{\text{V}} = 0. \tag{4.1}$$

Unlike equation (3.1), this is not a Sturm–Liouville eigenvalue/eigenvector problem, so that no general sufficient condition for instability can easily be established. In order to interpret this equation, we first draw an analogy with the standard case of a non-homogeneous flow under the action of gravity. For this, let us consider the Cartesian (x, y, z) setting shown in figure 1(*b*) which represents a stratified shear flow of velocity $\boldsymbol{u} = V(x)\boldsymbol{e}_y$ and density $R(x)$ with the gravity force $\boldsymbol{F} = -Rg\boldsymbol{e}_x$. This basic flow may undergo two-dimensional instability, i.e. the so-called RTI, if light fluid is below heavy fluid. If we compare figures 1(*a*) and 1(*b*), we see that the cylindrical and Cartesian problems are analogous, the centrifugal force playing the role of the gravity force. Note that if the grey levels represent high values of density, then the two sketched configurations are unstable: heavy fluid inside light fluid in a vortex is equivalent to light fluid below heavy fluid in a standard shear flow with gravity. In both cases, the unstable situation corresponds to the force (centrifugal or gravity) directed towards the light fluid.

In order to go deeper in the analogy, let us focus in the Cartesian setting on two-dimensional perturbations of the form $e^{-i\omega t}e^{ily}\psi(x)$ where ψ is the streamfunction perturbation, ω the complex frequency and l the wavenumber in the y-direction – we deliberately use the same notation as in the cylindrical setting. Yih (1965) showed that the linearization of the incompressible non-homogeneous Euler equations around this basic flow yields the following eigenvalue/eigenvector problem:

$$R^{-1}(R\psi')' - l^2\psi - \underbrace{l^2\Sigma^{-2}G^2\psi}_{\text{IV}} - \underbrace{l(R\Sigma)^{-1}(R\Xi)'\psi}_{\text{V}} = 0 \tag{4.2}$$

where the prime denotes differentiation with respect to the vertical coordinate x, $\Sigma = -\omega + lV$, $\Xi = V'$ is the vorticity of the basic flow and $G^2 = gR^{-1}R'$. Note that $-G^2$ corresponds to the square of the buoyancy frequency if $R' < 0$. If $V = 0$, equation (4.2) reduces to

$$R^{-1}(R\psi')' - l^2\psi - \underbrace{l^2\omega^{-2}G^2\psi}_{\text{IV}} = 0 \tag{4.3}$$

which yields the prototype RTI in the Cartesian setting (Rayleigh 1883). In these equations, term IV is responsible for the RTI and term V represents the action of shear, which may in particular produce KHI.

If we come back to the cylindrical problem, we can see that equation (4.2) is the same as equation (4.1) without curvature effects. Also, equation (4.3) is analogous, without curvature effects, to equation (4.4) which will be derived below from equation (4.1) in the asymptotic case $b \to 0$ and $C \to 0$, and which yields the prototype RTI in the cylindrical setting. This unambiguously shows that the instabilities described in the following are of the RTI type: they are produced by the centrifugal force which takes advantage of the flow inhomogeneities to destabilize the flow. Also, one may infer that term IV in equations (4.1) and (4.4) is responsible for the RTI whereas term V represents the action of shear. The latter may have two effects. First, as mentionned before and in the Introduction, it may be responsible for KHI. But, as the Lamb–Oseen azimuthal velocity profile does not present any extremum in the vorticity distribution for $r > 0$, we do not expect this instability to appear here. Secondly, and this is the main point, we will show below that shear has a stabilizing effect on RTI.

4.1. *Numerical results obtained with the shooting method for $k = 0$*

For all azimuthal wavenumbers $m \geqslant 1$, several unstable eigenmodes are generally obtained, and are labelled by the integer $n = 0, 1, 2,$ etc. which represents the number of nodes of the eigenfunction $\psi(r)$ in the radial direction. We first focus on the case $m = 3$. For this case, the most amplified eigenmode corresponds to $n = 0$. The corresponding amplification rate ω_i and oscillation rate ω_r are displayed, respectively, in figures 2(*c*) and 2(*d*). This particular eigenmode is stable for small density contrasts C and high density radii b. A non-trivial marginal stability boundary exists below which the eigenmode is unstable. In the unstable region, the amplification rate becomes stronger as the density contrast C increases: as will be shown in §4.3, the heavier the vortex, the faster the instability. The instability also becomes stronger as the density radius b decreases, which shows that maximum instability occurs when the density gradient is located in a region where the azimuthal velocity of the basic flow is shear-free, i.e. in a rigidly rotating flow. Hence, shear has a stabilizing effect on RTI. Note that the real part ω_r of the complex frequency plotted in figure 2(*d*) satisfies $0 < \omega_r/m < 1$. This is in accordance with various semi-circle theorems established by Lalas (1975), Warren (1975) and Fung (1983). This also shows that, on the marginal stability boundary, the neutral eigenmodes exhibit a critical layer at the radius r_c where $\Sigma(r_c) = 0$, i.e. $V(r_c)/r_c = \omega_r/m$.

For larger azimuthal wavenumbers ($m > 3$), the results are qualitatively similar to those presented above, and the most amplified eigenmode is always the primary one ($n = 0$). On the other hand, for $m = 1$ and $m = 2$, in some regions of the (C, b)-plane, higher-order eigenmodes ($n = 1, 2, \ldots$) were found to be more amplified than the primary one ($n = 0$). In figures 2(*e*) and 2(*f*), we have respectively sketched the iso-values ω_i and structure of the most unstable eigenmode over all azimuthal wavenumbers $m \geqslant 1$ and all $n \geqslant 0$. The amplification rate follows the same general trends as described for $m = 3$. Concerning the structure of the most amplified eigenmode, we notice that large-scale eigenmodes ($m = 1, n = 0, 1, 2$) and ($m = 2, n = 0$) are selected near the marginal stability boundary. When b decreases, eigenmodes become highly oscillatory in the azimuthal direction (high values of m) but remain large scale in the radial direction ($n = 0$). The area $b < 0.2$ has been left blank since the most unstable eigenmodes have very high azimuthal wavenumbers ($m \geqslant 8$) which have not been computed. In §4.3, a dedicated asymptotic analysis for $b \to 0$ will thoroughly explain the structure of the unstable eigenmodes in this area.

We can note that competition between CTI and RTI exists only for high values of C. In fact, for small values of C and b, the basic flow only has RTI whereas for small values of C and high values of b only CTI exists. In the case of high values of C, one has to carefully compare the numerical values of both amplification rates. In figures 2(*e*) and 2(*f*), we have therefore sketched a dotted line above which the flow is unstable to CTI eigenmodes whose amplification rates are higher than those of the RTI eigenmodes. This behavior may be explained in the following way. For the RTI, a necessary condition is $G^2 > 0$, which means that the density has to decrease with r somewhere in the flow. However this condition is not sufficient, except (see §4.3 for proof) in the case $C \to 0$ and $b \to 0$, where the density gradient is located in the centre of the vortex which is shear-free. For larger values of b, the shear has a stabilizing effect on the instability. On the other hand, for the CTI, a necessary and sufficient condition is $H^2 > 0$, which means as shown before that the quantity

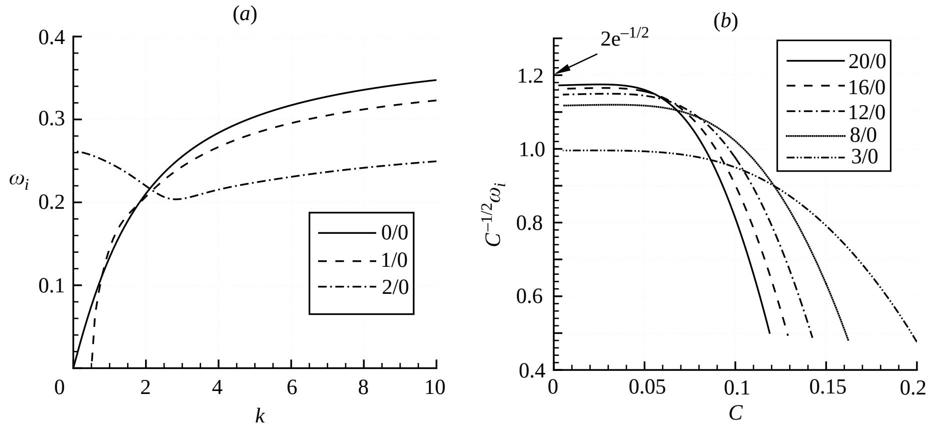

FIGURE 3. (a) Amplification rates ω_i as function of axial wavenumber k for various eigenmodes ($m=0, 1, 2, n=0$) in the case ($C=0.8, b=1$). (b) Amplification rates $C^{-1/2}\omega_i$ as a function of C along the dashed line in figure 2(c) for various eigenmodes ($m=3, 8, 12, 16, 20, n=0, k=0$).

Rr^2V^2 must decrease somewhere in the flow. The quantity r^2V^2 is strongly growing within the vortex core, but is nearly constant outside the vortex core, for $r \gtrsim 1$. This explains why for moderate density contrasts C, this instability only occurs for $b>1$, i.e. when the density gradient is located outside the vortex core. For small values of b, this instability may also take place, but a very large density contrast C is required for the density gradient to centrifugally destabilize the flow. As a result, the flow is strictly stable only for $C=0$ and a small region in the (C, b)-plane which roughly corresponds to the rectangle $0.8<b\leqslant 1$ and $0\leqslant C<0.5$.

4.2. *Numerical results obtained with the shooting method for $k \neq 0$*

Up to now we have investigated only pure two-dimensional ($k=0$), non-axisymmetric ($m\geqslant 1$) RTI. Here we study how the unstable modes evolve if we consider non-zero values of the axial wavenumber k. We distinguish different cases. If we choose C and b for which only RTI and no CTI exists, e.g. ($C=0.5, b=0.5$), then (not shown here) the amplification rates of all RTI eigenmodes are maximum for $k=0$, decrease with k and vanish for values of k of order 1. Note that an analogous result exists in the case for which only CTI and no RTI exists, e.g. ($C=0.3, b=1.5$). Therefore, we may conclude that three-dimensionality (respectively two-dimensionality) generally has a stabilizing effect on RTI (respectively CTI). The results are different if we choose C and b for which both RTI and CTI exist. For example, we show in figure 3(a) the amplification rate of various eigenmodes as a function of k for the case $C=0.8, b=1$. For $k=0$, the most amplified mode corresponds to ($m=2, n=0$), in accordance with figure 2(f). As k increases, it appears that the amplification rate first decreases, reaches a minimum for $k=3$, then increases again before converging towards a constant value as $k\rightarrow\infty$. In this limit, the most amplified mode is the axisymmetric CTI mode ($m=0, n=0$), and its amplification rate is in accordance with the asymptotic prediction (3.3). We have also considered the eigenmode corresponding to ($m=1, n=0$). For $k=0$, this eigenmode is attenuated. But, as k increases, this eigenmode becomes unstable and even more amplified than the ($m=2, n=0$) eigenmode for $k>2$.

These results indicate that, as k increases, the non-axisymmetric modes progressively change from an RTI nature to a CTI nature. This is consistent with the results of Gallaire & Billant (2003), who showed that in the homogeneous case, all eigenmodes are of the CTI type in the short-wavelength limit, the eigenmodes with $m \neq 0$ being sub-obtimal CTI eigenmodes. This conclusion is found to remain valid in the non homogeneous case.

4.3. *Asymptotic analysis of the case* $m \neq 0$, $k = 0$, $C \to 0$ *and* $b \to 0$

In this section, we perform an asymptotic stability analysis in the case where the RTI amplification rates are maximum, i.e. $b \to 0$. We have to restrict the analysis to the case $C \to 0$ to keep a tractable formalism. We will show that equation (4.1) reduces to a Sturm–Liouville problem which yields a sufficient condition for instability.

We introduce the parameter $\epsilon = C^{1/2}$ which is supposed to be small. All lengths are re-scaled by ϵ: $b = \epsilon\bar{b}$ and $r = \epsilon\bar{b}\bar{r}$. This means that the relevant length scale is no longer the azimuthal velocity length scale but instead the width of the heavy core b. The complex frequency of the eigenmode is composed of an oscillating part m and an amplification rate which scales on the density contrast ϵ: $\omega = m + \epsilon\bar{\omega}$. This means that the azimuthal phase velocity of the perturbation $\mathrm{d}\theta/\mathrm{d}t = \omega_r/m$ is equal to 1 which is the rotation rate V/r in the centre of the vortex. The real part of the complex frequency $\omega_r = m$ therefore reflects the convection of the perturbation by the rigidly rotating flow.

Introducing these scalings into (4.1), we obtain the following classical Sturm–Liouville eigenvalue/eigenvector problem at leading order in ϵ:

$$\bar{r}^{-1}(\bar{r}\psi')' - \bar{l}^2\psi - \underbrace{\bar{l}^2\bar{\omega}^{-2}\bar{G}^2\psi}_{\mathrm{IV}} = 0 \qquad (4.4)$$

with $\bar{l} = m\bar{r}^{-1}$ and the prime denoting differentiation with respect to $\bar{r}$. Note that a similar equation was obtained by Gans (1975) in the case of a strictly uniform rotation. Here $G^2 = \epsilon^2\bar{G}^2$ so that $\bar{G} = 2\bar{r}\exp(-\bar{r}^2/2)$. Comparing this equation to (4.1), we can see that the shear term V has disappeared. This stems from the facts that (i) the basic vorticity field Ξ becomes constant in the centre of the vortex and (ii) the density variations remain weak since G^2 is of order ϵ^2. From this, we may conclude that the flow field (2.1) is unstable for small values of C and b if $G^2 > 0$ somewhere.

In the limit of high azimuthal wavenumbers $m \gg 1$, we can construct analytical eigenvalues/eigenvectors localized radially in the vicinity of a maximum of the function $\bar{G}$. For this, we first note that the function $\bar{G}$ has a positive maximum at $\bar{r}_0 = 1$ with $\bar{G}(\bar{r}_0) = \bar{G}_0 = 2\mathrm{e}^{-1/2}$, $\bar{G}'(\bar{r}_0) = 0$, $\bar{G}''(\bar{r}_0) = \bar{G}_2 = -4\mathrm{e}^{-1/2}$. Then, we introduce the scaling $\bar{r} = \bar{r}_0(1 + \lambda^{1/2}m^{-1/2}\tilde{r})$ and $\bar{\omega} = \mathrm{i}\bar{G}_0[1 - (2\lambda)^{-1}m^{-1}\tilde{\omega}]$ with $\lambda = [-\bar{G}_0(4\bar{r}_0^2\bar{G}_2)^{-1}]^{1/2}$. If we let $m \to \infty$, equation (4.4) leads to the quantum harmonic oscillator (3.2) at leading order, which exhibits the following eigenvalues/eigenvectors:

$$\omega_n = m + \mathrm{i}\epsilon\bar{G}_0[1 - (2\lambda)^{-1}m^{-1}(n + 1/2)], \quad \psi_n = \mathrm{He}_n(\tilde{r})\exp(-\tilde{r}^2/4) \qquad (4.5)$$

To validate this asymptotic analysis, we show in figure 3(*b*) the re-scaled amplification rate $\epsilon^{-1}\omega_i = C^{-1/2}\omega_i$ obtained by the shooting method along the path sketched by a dashed line in figure 2(*c*). This path is characterized by the equation $\bar{b} = C/0.2$ or $b = C^{3/2}/0.2$. Several curves corresponding to various azimuthal wavenumbers $m = 3, 8, 12, 16, 20$ have been shown as a function of C in figure 3(*b*). We observe that all curves converge, as $C \to 0$, towards a constant value, which increases with m, and which tends towards the value $\bar{G}_0 = 2\mathrm{e}^{-1/2}$ predicted in (4.5) as $m \to \infty$.

This asymptotic analysis shows that for all $m \geqslant 1$, there exists an infinite number of unstable eigenmodes, labelled by the number of nodes n of the eigenmode, and whose amplification rate decreases with n and increases with m. These results are strictly valid only for $C \to 0$ and $b \to 0$. But as shown in §4.1, on the whole this structure remains valid for higher values of C and b. It is important to note that each unstable eigenmode found in § 4.1 and characterized by the azimuthal wavenumber m and the number of nodes n is continuously linked as $C \to 0$ and $b \to 0$ to the (m, n) eigenmode presented in (4.5). The action of shear represented by term V in equation (4.1) has a strong impact on the azimuthal structure of the most amplified eigenmode: whereas high azimuthal wavenumbers are favoured as $b \to 0$, only $m = 1$ and $m = 2$ eigenmodes appear near the marginal stability boundary in figure 2(f). This shows that the stabilizing effect of shear is more efficient in the case of high azimuthal wavenumbers.

REFERENCES

Bayly, B. 1988 Three-dimensional centrifugal-type instabilities in inviscid two-dimensional flows. *Phys. Fluids* **31**, 56–64.

Bender, C. & Orszag, S. 1978 *Advanced Mathematical Methods for Scientists and Engineers*. McGraw-Hill.

Coquart, L., Sipp, D. & Jacquin, L. 2004 Mixing induced by Rayleigh-Taylor instability in a vortex. Accepted for publication in *Phys. Fluids*.

Drazin, P. & Reid, W. 1981 *Hydrodynamic Stability*. Cambridge University Press.

Eckhoff, K. 1984 A note on the stability of columnar vortices. *J. Fluid Mech.* **145**, 417–421.

Fabre, D., Michelin, S., Sipp, D., Lombardini, E. & Jacquin, L. 2003 Linear stability of a vortex with heated or cooled core. *5th EUROMECH Fluid Mechanics Conference, August 24–28, 2003, Toulouse, France*.

Fung, Y. 1983 Non-axisymmetric instability of a rotating layer fluid. *J. Fluid Mech.* **127**, 83–90.

Gallaire, F. & Billant, P. 2003 Generalized Rayleigh criterium for non-axisymmetric centrifugal instabilities. In *56th Meeting of the APS Division of Fluid Dynamics, 23–25 November, 2003, East Rutherford (NJ), USA. Bull. Am. Phys. Soc.* **48**, 66.

Gans, R. 1975 On the stability of shear flow in a rotating gas. *J. Fluid Mech.* **68**, 403–412.

Howard, L. 1961 Note on a paper of John W. Miles. *J. Fluid Mech.* **10**, 509–512.

Howard, L. 1973 On the stability of compressible swirling flow. *Stud. Appl. Maths* **52**, 39–43.

Joly, L., Fontane, J. & Chassaing, P. 2004 The baroclinic centrifugal instability of variable-density two-dimensional vortices. Submitted to *J. Fluid Mech.*

Lalas, D. 1975 The 'Richardson' criterion for compressible swirling flows. *J. Fluid Mech.* **69**, 65–72.

Le Duc, A. 2001 Étude d'écoulements faiblement compressibles, de giration, puis d'impact sur paroi, par théorie linéaire et simulation numérique directe. Thèse de doctorat, Ecole Centrale de Lyon, France.

Le Duc, A. & Leblanc, S. 1999 A note on Rayleigh stability criterion for compressible flows. *Phys. Fluids* **11**, 3563–3566.

Leibovich, S. 1969 Stability of density stratified rotating flows. *AIAA J.* **7**, 177–178.

Lin, C. 1955 *The Theory of Hydrodynamic Stability*. Cambridge University Press.

Rayleigh, Lord 1883 Investigation of the character of the equilibrium of an incompressible fluid with variable density. *Proc. Lond. Math. Soc.* **14**, 170–177.

Warren, F. 1975 A comment on Gans' stability criterion. *J. Fluid Mech.* **68**, 413–415.

Yih, C.-S. 1965 *Dynamics of Nonhomogeneous Flows*. Macmillan.